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COMMENT

Action and pseudocharge of an electromagnetic wave

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Abstract. The action s and pseudocharge q of an electromagnetic wave are defined by $2s = \int (\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B})\, \mathrm{d}^4x$ and $q = -\int (\boldsymbol{E}\cdot\boldsymbol{B})\, \mathrm{d}^4x$. Both s and q are shown to vanish for an electromagnetic wave (in free space) which is spatially bounded at some time $t = 0$.

The action s and pseudocharge q of an electromagnetic wave can be defined by

$$s = \tfrac{1}{2}\int (\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B})\, \mathrm{d}^4x \tag{1}$$

and

$$q = -\int (\boldsymbol{E}\cdot\boldsymbol{B})\, \mathrm{d}^4x \tag{2}$$

in units where $c = 1$. Recently, Khare and Pradhan (1982a) constructed a free-space electromagnetic wave (based on an example of Chu and Ohkawa (1982)) which had non-zero s and non-zero q. Although their development was technically flawed and subsequently altered, Khare and Pradhan (1982b, 1983) maintained their assertion of the existence of a free-space electromagnetic wave with non-zero s and q.

The essence of the example of Khare and Pradhan hinges upon the $k = 0$ behaviour of the Fourier decomposition of their vector potential $\boldsymbol{A}(\boldsymbol{x}, t)$. Their Fourier coefficient $C(k)$ is singular in the manner of $1/k^2$ near $k = 0$. As pointed out by Michel (1984), this corresponds to simply superposing uniform static $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

Thus the question seemingly remains open: does there exist a bona fide example of an electromagnetic wave in free space which has non-zero s or q? The purpose of this comment is to answer the question in the negative.

Let $\boldsymbol{E}(\boldsymbol{x}, t)$ and $\boldsymbol{B}(\boldsymbol{x}, t)$ be solutions of the free-space Maxwell equations. We shall assume that these fields are spatially bounded at $t = 0$ (and hence at all finite times); note that this does not imply that the vector potential $\boldsymbol{A}(\boldsymbol{x}, t)$ is spatially bounded. As for the integrations in (1) and (2) we may, in view of the spatially bounded fields, regard the $\mathrm{d}\tau = \mathrm{d}^3x$ integration as being over all space and the $\mathrm{d}t$ integration as running from $t = T_1$ to $t = T_2$ with the limits $T_1 \to -\infty$ and $T_2 \to +\infty$ applied last.

We choose a gauge in which the scalar potential vanishes; thus $\boldsymbol{E} = -\partial \boldsymbol{A}/\partial t$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. Using these relations and the Maxwell equation $\nabla \times \boldsymbol{B} = \partial \boldsymbol{E}/\partial t$, the following identity is readily established:

$$\boldsymbol{E}\cdot\boldsymbol{E} - \boldsymbol{B}\cdot\boldsymbol{B} = -\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) + \frac{1}{2}\frac{\partial^2}{\partial t^2}(\boldsymbol{A}\cdot\boldsymbol{A}). \tag{3}$$

When this is substituted into (1), the divergence term integrates to zero in view of the fact that $\boldsymbol{B}$ (though not necessarily $\boldsymbol{A}$) is spatially bounded. The action s becomes

$$s=\frac{1}{4}\int\int\frac{\partial^2}{\partial t^2}(\boldsymbol{A}\cdot\boldsymbol{A})\,\mathrm{d}\tau\,\mathrm{d}t=M(T_2)-M(T_1) \tag{4}$$

where

$$M(t)=\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\int(\boldsymbol{A}\cdot\boldsymbol{A})\,\mathrm{d}\tau. \tag{5}$$

For s to have a well-defined meaning we must assume that the right-hand side of (4) is independent of T_1 (provided it is large and negative) and independent of T_2 (provided it is large and positive). Since $\int(\boldsymbol{A}\cdot\boldsymbol{A})\,\mathrm{d}\tau>0$ we must have that $M(T_2)\geqslant 0$ for otherwise $\int(\boldsymbol{A}\cdot\boldsymbol{A})\,\mathrm{d}\tau$ would become negative for some $t>T_2$. A similar argument shows that $M(T_1)\leqslant 0$. From (4) we then have that

$$s\geqslant 0. \tag{6}$$

Now consider a duality transformation with 'rotation parameter' θ:

$$\boldsymbol{E}'=\boldsymbol{E}\cos(\theta)-\boldsymbol{B}\sin(\theta) \tag{7a}$$

$$\boldsymbol{B}'=\boldsymbol{B}\cos(\theta)+\boldsymbol{E}\sin(\theta). \tag{7b}$$

This preserves the Maxwell equations as well as the spatial boundedness of the fields. Using (1), (2) and (7) one finds that

$$s'=s\cos(2\theta)+q\sin(2\theta) \tag{8a}$$

$$q'=q\cos(2\theta)-s\sin(2\theta). \tag{8b}$$

Choosing $\theta=\frac{1}{2}\pi$ yields

$$s'=-s\leqslant 0. \tag{9}$$

Equation (6), however, applies equally well to s'. A contradiction then ensues unless $s=s'=0$. With $s=s'=0$, a similar argument using $\theta=\frac{1}{4}\pi$ leads to $q=q'=0$.

In summary, we have shown that if the electromagnetic fields ($\boldsymbol{E}$ and $\boldsymbol{B}$) are spatially bounded and if the expressions in (1) and (2) have a well-defined meaning, then both s and q must be zero.

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